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CONCORDANCE IS EQUIVALENT TO SMOOTHABILITY

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LET $h: K \to M$ and $k: K \to N$ be smooth (C^∞) triangulations of the non-bounded manifolds M and N. Hirsch defines the differentiable structures α and β on K induced by h and k, respectively, to be *concordant* if there is a differentiable structure γ on $K \times I$ (compatible with its piecewise-linear structure) whose restrictions to $K \times 0$ and $K \times 1$ equal α and β, respectively. He has outlined in [1] a theory of obstructions to the existence of a concordance γ.

It follows from [3] that concordance of α and β is equivalent to the existence of a *combinatorial deformation* between the combinatorial equivalence $f = kh^{-1}: M \to N$ and a diffeomorphism. This means that there exist piecewise-smooth triangulations (not necessarily level preserving)

$$\tilde{h}: K \times I \to M \times I \qquad \text{and} \qquad \tilde{k}: K \times I \to N \times I$$

whose restrictions to $K \times 0$ equal h and k, respectively, such that $\tilde{k}\tilde{h}^{-1} \mid (M \times 1)$ is a diffeomorphism. (A combinatorial equivalence is, strictly speaking, not a map f but a triple (K,h,k); referring to f as a combinatorial equivalence is an abuse of notation we sometimes find convenient.)

On the other hand, we have defined a notion of when f is *smoothable* to a diffeomorphism; this notion is *a priori* weaker than the requirement that f be combinatorially deformable to a diffeomorphism. (See 3.10 of [5].) We have constructed a theory of first-order and higher-order obstructions to the existence of a smoothing of f [2, 5]. Each of the two obstruction theories seems to have its unique insights, so that settling the differences between them has become an issue.

Our purpose here is to prove the following:

THEOREM 2.2. *If f may be smoothed to a diffeomorphism mod L, where $h^{-1}(L)$ is a subcomplex of K, then given any neighborhood U of L, f may be combinatorily deformed to a combinatorial equivalence $g: M \to N$ which is a diffeomorphism outside U. The converse holds as well; the theorem also holds if the combinatorial deformation is required to be strong (i.e. level-preserving).*

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At first glance, one might wonder whether g could be constructed to be a diffeomorphism on $M - L$ rather than on $M - U$. This is in fact impossible, for elementary arguments show that any function which is piecewise-differentiable on M and differentiable on $M - L$ is necessarily differentiable on all of M.

The proof of the theorem involves the notion of a *smooth cell decomposition* of a manifold M. This is defined to be a regular cell complex whose topological space is M, such that each closed m-cell is *smooth* in the sense that it lies in a smooth open m-dimensional submanifold of M; these submanifolds are to intersect transversally (in the sense of 5.3 of [5] or 10.10 of [6]). If $h : K \to M$ is a smooth triangulation, the simplicial decomposition of K induces a smooth cell decomposition of M, but the standard dual cell decomposition of K does not. We prove in 1.4 that it is possible to modify h so as to obtain a smooth triangulation $h' : K \to M$ under which the dual cells do induce a smooth cell decomposition of M. In fact, we construct h' so that each set $h'(\bar{\tau})$ intersects each set $h'(\sigma)$ orthogonally, with respect to suitably chosen coordinate systems on M; here σ is the general simplex of K and τ is the general dual cell. Furthermore, we also construct a *smooth isotopy* h_t connecting h and h'; this is a map $h_t : K \times I \to M$ such that h_t is a smooth triangulation for each t, h_t is smooth on each set $\bar{\sigma} \times I$, and $h_0 = h$ and $h_1 = h'$. In addition, it happens that h_t satisfies the condition $h_t(\sigma) = h(\sigma)$ for each simplex σ of K and each t. The construction is carried out in §1; it requires considerable care but is elementary in nature; it seems to us of some independent interest.

The new triangulation h' has the following advantage for smoothing purposes: In general, if we take the combinatorial equivalence f and smooth it to a diffeomorphism mod L, the smoothed map f_L will not preserve the simplicial structures of M and N induced by h and k; in fact, the sets $f_L h(\bar{\sigma})$ will not even be smooth cells in N. Suppose however that we replace h and k by the special triangulations h' and k' and start to smooth $f' = k'(h')^{-1}$. We find that at each stage the smoothed map may be chosen to carry the smooth cell $h'(\tau)$ onto the smooth cell $k'(\tau)$, so that it preserves the new dual cell structures of M and N (but not the new simplicial structures, of course). This result is proved in 2.1; one uses the fact that the dual cells are not only smooth but also orthogonal to the simplices they intersect.

The theorem follows readily; we sketch its proof in the case where L is empty. Since f is combinatorially deformable to f', and f may be smoothed to a diffeomorphism, f' may also be smoothed to a diffeomorphism, say g (by 2.5 and 3.2 of [5]). As just noted, we may choose g so that it carries each dual cell $h'(\tau)$ onto $k'(\tau)$. Consider the two smooth triangulations $h' : K \to M$ and $g^{-1}k' : K \to M$. Using techniques of J. H. C. Whitehead, we can find a subdivision K' of K and a smooth triangulation $h'' : K' \to M$ such that the composite $\phi = (g^{-1}k')^{-1}h'' : K \to K$ is piecewise linear, such that h'' is smoothly isotopic to $h' : K' \to M$, and such that $h'(\tau) = h''(\tau)$ for each dual cell τ. Here we use crucially the fact that the dual cells are smooth. (See 10.9 and 10.12 of [6].) It is easy to show that ϕ is piecewise linearly isotopic to the identity, using the Alexander construction for deforming a piecewise linear homeomorphism of a combinatorial ball into the identity map. The theorem follows: f is combinatorially deformable to f'; $f' = k'(h')^{-1}$ is combinatorially deformable to $k'(h'')^{-1}$; and (since ϕ is piecewise linearly isotopic to the identity) $k'(h'')^{-1}$ is combinatorially deformable to $k'\phi(h'')^{-1} = g$.

As one consequence of the theorem, we have the following:

COROLLARY 2.4. *The following statements are equivalent*:

(0) *There exists a diffeomorphism* $g : M \to N$ *carrying each dual cell* $h'(\tau)$ *onto* $k'(\tau)$.

(1) f *is smoothable to a diffeomorphism.*

(1a) f *is smoothable to a diffeomorphism* g *satisfying* (0).

(2) f *is combinatorially deformable to a diffeomorphism.*

(2a) f *is strongly combinatorially deformable to a diffeomorphism* g *satisfying* (0); *the deformation* $(\tilde{h}, \tilde{k})$ *may be chosen so that* $\tilde{h}$ *is the trivial extension* $h \times i$ *of* h.

(3) α *is concordant to* β.

(3a) *There is a concordance* γ *between* α *and* β *such that each slice* $K \times t$ *is a differentiable submanifold of* $(K \times I)_\gamma$.

Proof. Consider the following chains of implications: (2a) $\Rightarrow$ (1a) $\Rightarrow$ (1) and (2a) $\Rightarrow$ (3a) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (0) $\Rightarrow$ (2a). It is a consequence of the definition that (2a) $\Rightarrow$ (1a) and (2) $\Rightarrow$ (1): If $F : M \times I \to N \times I$ is a combinatorial deformation between f and g, we may extend F trivially to $G : M \times R \to N \times R$. Then G is a deformation mod $M \times (-\infty, 1]$ between f and g, so that g is a smoothing of f, by Definition 1.4 of [5].

The implications (1a) $\Rightarrow$ (1) and (3a) $\Rightarrow$ (3) are clear; and (2a) $\Rightarrow$ (3a) follows from the definition: Let $(h \times i, \tilde{k})$ be the strong combinatorial deformation. The differentiable structure γ on $K \times I$ induced by $\tilde{k} : K \times I \to N \times I$ is the desired concordance; the induced structure on $K \times 0$ is β, and on $K \times 1$ it is α. Since the combinatorial deformation is strong, each slice $K \times t$ is a differentiable submanifold of $(K \times I)_\gamma$.

The implication (3) $\Rightarrow$ (2), as mentioned above, follows from [3]: Let γ be the concordance between α and β. Let $F : (K \times I)_\gamma \to K_\alpha \times I$ be induced by the identity map; then F is a combinatorial equivalence which is a diffeomorphism on $(K \times 0)_\gamma$. By Theorem 2 of [3], there is a diffeomorphism $G : (K \times I)_\gamma \to K_\alpha \times I$ which equals the identity on $(K \times 0)_\gamma$. $(h \times i)Gj : K \times I \to M \times I$ and $(k \times i) : K \times I \to N \times I$ is easily seen to be the desired combinatorial deformation, where $j : K \times I \to (K \times I)_\gamma$ is the identity triangulation.

The two remaining implications depend on the main theorems of the present paper, 2.1 and 2.2 respectively; their proof is saved for later.

The equivalence of concordance and smoothability shows that all obstructions vanish in Hirsch's theory if and only if they vanish in ours. One may well ask what happens more generally. At the pth stage in our obstruction theory, we have a smoothing $f_m : M \to N$ of f which is a diffeomorphism mod the m-skeleton $L = h(K^m)$ of M (where $m = n - p$); the obstruction to the next step is an element of $\mathscr{H}_m(M; \Gamma_p)$. At the pth stage of Hirsch's theory, he has a differentiable structure γ on a neighborhood in $K \times I$ of $(K \times 0) \cup (K^{p-1} \times I) \cup (K \times 1)$; the obstruction to the next step lies in $H^p(M; \Gamma_p)$. The isomorphism $\mathscr{H}_m(M; \Gamma_p) \cong H^p(M; \Gamma_p)$ suggests that these obstructions should be dual to one another. This we do not specifically prove, partly because of the technical complications involved and partly because Hirsch informs us that he intends to use yet another definition of his obstruction cochain in the detailed version of [1]. We content ourselves with showing that the first p obstructions vanish in one theory if and only if they vanish in the other:

COROLLARY 2.5. *The following conditions are equivalent*:

(1) *f may be smoothed to a diffeomorphism mod the m-skeleton $h(K^m)$ of M.*

(2) *The differentiable structures α and β on $K \times 0$ and $K \times 1$, respectively, may be extended to a differentiable structure on a neighborhood in $K \times I$ of $(K \times 0) \cup (K^{p-1} \times I) \cup (K \times 1)$.*

The implication (1) $\Rightarrow$ (2) follows from results of the present paper; the converse depends on the relative version of Hirsch's basic theorem (3.1 of [1]).

A final corollary of our theorem has to do with the problem of imposing a compatible differentiable structure on a combinatorial manifold. It follows from the present paper that any differentiable structures on a combinatorial manifold K which are constructed by means of the author's obstruction theory [4] will in fact be compatible with the piecewise linear structure of K. We intend to treat this elsewhere.

§1. SMOOTH DUAL CELL DECOMPOSITIONS OF MANIFOLDS

Throughout the paper, M will denote a differentiable manifold of dimension n which is non-bounded (i.e. without boundary), and $h : K \to M$ will denote a smooth (C^∞) triangulation of M. The general (open) simplex of K will be denoted by σ.

Definition 1.1. Let K' be the first barycentric subdivision of K. If σ is an m-simplex of K, the *dual cell* $\tau(\sigma)$ is the union of all simplices of K' whose closures intersect σ in precisely the barycenter $\hat{\sigma}$ of σ; $\bar{\tau}(\sigma)$ is a combinatorial ball of dimension $n - m$.

We define $\tau(\sigma)$ to be a *face* of $\tau(\sigma_0)$ if σ_0 is a face of σ; the collection of dual cells then defines a regular cell decomposition of $|K|$.

Definition 1.2. Given σ, let us number the vertices of K so that $\sigma = v_0 \cdots v_m$. Then $\bar{\tau}(\sigma)$ consists of all points b such that $b_0 = b_1 = \cdots = b_m$, and $b_0 \geq b_j$ for all other j, where the b_i are the barycentric coordinates of b. Given $0 < \varepsilon < 1$, let $c_\varepsilon(\sigma)$ consist of all points b such that $b_0 = b_1 = \cdots = b_m$ and $b_0 > (1 - \varepsilon)b_j$ for all other j. Then $c_\varepsilon(\sigma)$ is an (open) cell of dimension $n - m$ containing $\tau(\sigma)$ and contained in the join $\hat{\sigma} * Lk\sigma$. We will call the cells c_ε *transverse cells*, for convenience; $c_\varepsilon(\sigma)$ is said to be *transverse* to the simplex σ. We note that a transverse cell $c_\varepsilon(\sigma_0)$ intersects a simplex σ only if σ_0 is a face of σ; in this case, the intersection is an (open) rectilinear cell whose dimension equals dim σ − dim σ_0. [It is helpful in visualizing this situation to let K be a 3-manifold, σ a 2-simplex, and σ_0 one of its edges.]

Definition 1.3. Let $u : U \to R^n$ be a coordinate system about $h(\bar{\sigma})$ such that $uh(\bar{\sigma})$ is a closed rectilinear simplex $\bar{s}$ in R^n, and uh carries each face of σ onto a face of s (but uh is not necessarily linear). In this case, we will say simply that *$\bar{\sigma}$ is rectilinear under uh.*

Let c_ε be a transverse cell in K; let $\bar{\sigma}$ be rectilinear under uh; let x be a point of $c_\varepsilon \cap \sigma$. We say that c_ε is *locally orthogonal to σ at x under uh* if the following holds:

There is a neighborhood U_x of x in c_ε such that π carries $uh(U_x)$ into $uh\,(c_\varepsilon \cap \sigma)$, where π is the orthogonal projection of R^n onto the plane of $uh(\sigma)$.

If this condition holds for each x in $c_\varepsilon \cap \sigma$, we say simply that c_ε is *orthogonal to σ under*

uh. Note that if $c_\varepsilon(\sigma_0)$ is locally orthogonal at x to σ under uh, so is $c_\delta(\sigma_0)$ (provided it contains x).

THEOREM 1.4. *Let $h: K \to M$ be a smooth triangulation of the non-bounded n-manifold M. There exist a smooth triangulation $h': K \to M$, and coordinate systems u_s about $h'(\bar{s})$ for each n-simplex s of K, such that*

(1) *$\bar{s}$ is rectilinear under $u_s h'$.*

(2) *For some $\varepsilon > 0$, each transverse cell c_ε is orthogonal to σ under $u_s h'$, where s is any n-simplex of K and σ is any face of s.*

(3) *There is a smooth isotopy h_t connecting h and h' such that $h_t(\sigma) = h(\sigma)$ for each t and each simplex σ of K.*

Proof. The proof proceeds by induction. We begin by choosing a coordinate system u_s about $h(\bar{s})$ satisfying (1), for each s. At the mth step of the induction, we assume that h and u_s are given such that $\bar{s}$ is rectilinear under $u_s h$, and such that for each simplex σ for which $\dim \sigma < m$, each coordinate system u_s (for which s has σ as a face), and each transverse cell c_ε, c_ε is orthogonal to σ under $u_s h$. (Initially, this condition is trivially satisfied for $m = 1$.)

Step 1. We apply Lemma 1.7, which follows. Given the m-simplex σ, choose an n-simplex $s(\sigma)$ having σ as a face; denote the corresponding coordinate system $u_{s(\sigma)}$ by u_σ. Choose $\delta < \varepsilon$; then alter h to h' by a smooth isotopy, so that each transverse cell c_δ is orthogonal to σ under $u_\sigma h'$. Since $h(\sigma_0) = h'(\sigma_0)$ for every simplex σ_0, each $\bar{s}$ is still rectilinear under $u_s h'$. Since h' is equal to h within a neighborhood of the $m - 1$ skeleton of K, the induction hypothesis is still satisfied if we replace h by h'. Since h' equals h outside a neighborhood of the open simplex σ, we may carry out this construction for each m-simplex independently. Let the resulting new triangulation be denoted by h''. So far we have altered the triangulation only; now we alter the coordinate systems.

Step 2. Now we apply Lemma 1.8. Let s be an n-simplex of K; let σ be an m-dimensional face of s; let $\gamma < \delta$. Using the fact that each c_δ is orthogonal to σ under $u_\sigma h''$, we may alter u_s to a coordinate system u_s' so that each c_γ is orthogonal to σ under $u_s' h''$. (No alteration of u_s is required, of course, if $s = s(\sigma)$.) Since $u_s' h''(\sigma_0) = u_s h''(\sigma_0)$ for each face σ_0 of s, $\bar{s}$ is still rectilinear under $u_s' h''$. Since u_s' is equal to u_s in a neighborhood of $h''(K^{(m-1)})$, each transverse cell c_δ is still orthogonal to each face of s of dimension less than m under $u_s' h''$. Finally, since u_s' is equal to u_s outside a neighborhood of the open simplex $h''(\sigma)$, we may carry out this construction independently for each m-face σ of s. Let the resulting coordinate system be denoted by u_s''. The induction hypothesis is then satisfied with m replaced by $m + 1$, h by h'', ε by γ, and u_s by u_s'' for each s.

The theorem follows by induction.

THEOREM 1.5. *If $h': K \to M$ is a smooth triangulation satisfying* (1) *and* (2) *of the preceding theorem, the images $h'(\tau)$ of the dual cells of K form a smooth cell decomposition of M.* (The converse is probably true also, but we shall not prove it.)

Proof. Let u_s be as in 1.4. For each τ, the corresponding transverse cell c_ε is an open cell containing τ; let $N_\tau = h'(c_\varepsilon)$. We need only prove that the sets N_τ are smooth submanifolds of M and that they intersect transversally.

Let x be a point of an m-simplex σ of K. Now K lies in some R^q; we may suppose $\bar{\sigma}$ lies in the plane $R^m \times 0$ of R^q. Choose an n-simplex s of which σ is a face, so that u_s is a coordinate system about $h'(\bar{\sigma})$; we may suppose that $u_s h'(\bar{\sigma})$ lies in the plane $R^m \times 0$ of R^n. Let p_s be the trivial extension of $u_s h' \mid \bar{\sigma}$ to a diffeomorphism of $\bar{\sigma} \times R^{n-m}$ into $R^m \times R^{n-m}$. Then $p_s^{-1} u_s$, restricted to a neighborhood of $h'(x)$, is a coordinate system about $h'(x)$, and the composite $p_s^{-1} u_s h'$ is the identity map of $\bar{\sigma} \times 0$ onto itself. We show there is a neighborhood V of $h'(x)$ in M such that for each N_τ containing $h'(x)$, the coordinate system $p_s^{-1} u_s$ carries $V \cap N_\tau$ onto an open subset of a plane in R^n. This proves 1.5.

Let c_ε be a transverse cell of dimension r, containing x. By hypothesis, there is a neighborhood U_x of x in c_ε such that orthogonal projection π of R^n onto $R^m \times 0$ carries $u_s h'(U_x)$ into $u_s h'(c_\varepsilon \cap \sigma)$. Then $p_s^{-1} u_s h'(U_x)$ must lie in $(c_\varepsilon \cap \sigma) \times R^{n-m}$. Since $c_\varepsilon \cap \sigma$ is a rectilinear cell of dimension $m - n + r$, $p_s^{-1} u_s$ carries $h'(U_x)$ into a plane of dimension r in R^n; since U_x is open in the r-cell c_ε, the image is open in this plane.

There is one such neighborhood U_x for each transverse cell c_ε containing x. Choose a neighborhood V of $h'(x)$ small enough that for each such c_ε, $(h')^{-1}(V) \cap c_\varepsilon$ lies in the corresponding neighborhood U_x. This choice of V satisfies our requirements.

The geometric constructions underlying the proof of Theorem 1.4.

LEMMA 1.6. *Let $h : K \to M$ be a smooth triangulation; let $u : U \to R^n$ be a coordinate system such that $\bar{\sigma}$ is rectilinear under uh. Let σ_0 be a face of σ. If c_ε is a transverse cell which is orthogonal under uh to σ_0, and if $\delta < \varepsilon$, then c_δ is locally orthogonal under uh to σ, at points of σ sufficiently near σ_0 (i.e., lying in some neighborhood W of σ_0 in $|K|$).*

Proof. Let dim $\sigma = m$; let dim $\sigma_0 = m_0$. Let $g = uh$. For each x in $g(c_\varepsilon \cap \sigma_0)$, let P_x denote the $n - m_0$ plane orthogonal to $g(\sigma_0)$ at x. By hypothesis, there is a neighborhood U_0 of $c_\varepsilon \cap \sigma_0$ in c_ε such that $g(U_0)$ lies in the space which is the union of the planes P_x. In fact $g(U_0)$ is open in this space, since this space is a non-bounded manifold whose dimension is that of c_ε. Hence we may assume that U_0 is so chosen that $g(U_0)$ intersects each plane P_x in a spherical neighborhood B_x of x in P_x. We shall prove that c_ε is locally orthogonal to σ under g at each point of $c_\varepsilon \cap \sigma$ lying in U_0. Then since $\bar{c}_\delta \subset c_\varepsilon$, we may choose a neighborhood W of σ_0 such that $W \cap c_\delta \cap \sigma$ lies in U_0; it follows that W satisfies the demands of the lemma.

Let π denote orthogonal projection of R^n onto the plane of $g(\sigma)$. Let U consist of those points y of U_0 such that $\pi g(y)$ lies in $g(\sigma)$. Then U is open in U_0 and contains $U_0 \cap c_\varepsilon \cap \sigma$; it will suffice to prove that $\pi g(U)$ lies in $g(c_\varepsilon \cap \sigma)$. Let y belong to U. Then $g(y)$ lies in $g(U_0)$ and hence in some spherical neighborhood B_x. Let $z = \pi g(y)$; z lies in $g(\sigma)$ by hypothesis. Since the triangle joining $g(y)$, z, and x has a right angle at z, z necessarily lies in B_x; thus $z = \pi g(y)$ lies in $g(U_0)$ also. Hence $\pi g(y)$ lies in $g(U_0 \cap \sigma)$, which is contained in $g(c_\varepsilon \cap \sigma)$.

LEMMA 1.7. *Let $h : K \to M$ be a smooth triangulation. Suppose that for each m-simplex σ of K there is a coordinate system u_σ about $h(\bar{\sigma})$ such that $\bar{\sigma}$ is rectilinear under $u_\sigma h$, and such that each transverse cell c_ε of K is orthogonal under $u_\sigma h$ to each proper face of σ. Given $\delta < \varepsilon$, there is a smooth triangulation $h' : K \to M$ smoothly isotopic to h such that each c_δ is orthogonal* *under $u_\sigma h'$ to σ. The map h' equals h in a neighborhood of the $m-1$ skeleton of K and outside a given neighborhood of the open simplex σ. If h_t is the isotopy, $h_t(\sigma_0) = h(\sigma_0)$ for every simplex σ_0 of K, and each t.*

Proof. We shall construct a homeomorphism f' of $|K|$ onto itself which carries each closed simplex of K diffeomorphically onto itself; given a neighborhood of the open simplex σ, we will choose f' so that it equals the identity outside this neighborhood and inside some neighborhood of $Bd\,\sigma$. The triangulation h' will be defined as the composite hf'.

Let B^p be the complex which is the join of $Lk\,\sigma$ with a point q. $(p = n - m)$. If we take $\bar\sigma \times B^p$ and identify each set $\bar\sigma \times y$ to a point, where y is in $Lk\,\sigma = Bd\,B^p$, we obtain a homeomorph of $\overline{St}\,\sigma$. More specifically, let $0 \le t \le 1$, let $x \in \bar\sigma$, and let $y \in Lk\,\sigma$. Define $\phi : \bar\sigma \times B^p \to \overline{St}\,\sigma$ by the equation

$$\phi(x, tq + (1-t)y) = tx + (1-t)y.$$

Then ϕ induces the described decomposition of $\bar\sigma \times B^p$. More than that, $\bar\sigma \times B^p$ is a rectilinear cell complex, ϕ carries each cell differentiably onto a simplex of $\overline{St}\,\sigma$, and ϕ^{-1} is differentiable on $\bar s - Lk\,\sigma$, for each n-simplex s of $\overline{St}\,\sigma$. It will be convenient for us to obtain f' by defining first a homeomorphism f of $\bar\sigma \times B^p$ with itself and then letting f' equal $\phi f \phi^{-1}$.

Henceforth we assume that B^p lies in R^p and is the join of the origin with a combinatorial $p-1$ sphere, and that $\bar\sigma$ is a rectilinear simplex in R^m. Then $\bar\sigma \times B^p \subset R^m \times R^p = R^n$. We also assume that $u_\sigma h(\sigma)$ lies in $R^m \times 0 \subset R^n$.

Let g be the following composite, defined in a neighbourhood of $\bar\sigma \times 0$:

$$\bar\sigma \times B^p \xrightarrow{\phi} \overline{St}\,\sigma \xrightarrow{h} M \xrightarrow{u_\sigma} R^n.$$

The restriction of g to $\bar\sigma \times 0$ induces a map of $\bar\sigma \to R^m$ which we denote by g_0.

We now translate the local orthogonality part of the hypothesis into a condition on the map g. It is easily checked that the sets $c_\varepsilon \cap St\,\sigma$ and $(c_\varepsilon \cap \sigma) * Lk\,\sigma$ coincide within some neighborhood of σ, for all transverse cells c_ε. This implies that the sets $\phi^{-1}(c_\varepsilon \cap St\,\sigma)$ and $\phi^{-1}((c_\varepsilon \cap \sigma) * Lk\,\sigma)$ coincide within a neighborhood of $\sigma \times 0$ in $\sigma \times B^p$. The latter set equals $(c_\varepsilon \cap \sigma) \times B^p$. As a result, the condition that c_ε be locally orthogonal to σ under uh at the point x of $c_\varepsilon \cap \sigma$ is equivalent to the following condition:

(*) *There is a neighborhood V of $(x, 0)$ in $(c_\varepsilon \cap \sigma) \times B^p$ such that $g(V)$ lies in $g_0(c_\varepsilon \cap \sigma) \times R^p$.*

By the hypothesis of the lemma and 1.6, we know there is a neighborhood W_1 of $Bd\,\sigma$ such that this condition holds with c_ε replaced by c_δ, for each transverse cell c_δ and each point x of $c_\delta \cap \sigma$ lying within W_1.

We will construct the homeomorphism f of $\bar\sigma \times B^p$ onto itself to satisfy the following conditions:

(a) *f equals the identity outside a given neighborhood N of $\sigma \times 0$ and inside some neighborhood of $Bd\,\sigma \times 0$, and on $\sigma \times 0$ itself.*

(b) *f is smoothly isotopic to the identity; the isotopy f_t is a diffeomorphism of each closed cell of $\bar\sigma \times B^p$ onto itself for each t.*

(c) *For any transverse cell* c_δ and any x in $c_\delta \cap \sigma$, *there is a neighborhood* V *of* $(x, 0)$ *in* $(c_\delta \cap \sigma) \times B^p$ *such that* $gf(V)$ *lies in* $g_0(c_\delta \cap \sigma) \times R^p$.

It is easily checked that this choice of f will suffice to prove our lemma.

Construction of f. Let us express the map g in coordinates by the equation

$$g(x,z) = (X(x,z), Z(x,z)),$$

where x and X are in R^m, and z and Z are in R^p. If z is in B^p, let us define $\theta_z : \bar{\sigma} \to R^m$ by the equation $\theta_z(x) = X(x,z)$. Since $\partial X/\partial x(x,z)$ is continuous, θ_z is a good C^1 approximation to θ_0 when z is small. Hence θ_z is a diffeomorphism if z is small, say $\|z\| < \delta_1$; $\theta_0 = g_0$, of course.

Let us define $\chi(x,z) = \theta_z^{-1}\theta_0(x)$ whenever $\|z\| < \delta_1$ and $\theta_0(x)$ lies in $\theta_z(\sigma)$. The map χ has the following properties:

(i) $X(\chi(x,z),z) = X(x,0)$.

(ii) $\chi(x,0) = x$.

(iii) $\chi(x,z)$ is a C^∞ function on the intersection of its domain with each closed cell of $\bar{\sigma} \times B^p$.

Property (i) follows from the computation $X(\chi(x,z),z) = \theta_z(\chi(x,z)) = \theta_0(x) = X(x,0)$. Property (ii) is trivial; property (iii) follows once we note that the map $(x,z) \to (\chi(x,z),z)$ is the composite of $(x,z) \to (X(0,z),z)$ and the inverse of $(x,z) \to (X(x,z),z)$.

Choose a neighborhood W_2 of $Bd\,\sigma$ such that $\overline{W}_2 \subset W_1$; choose ε_0 so that (x,z) lies in N whenever x is in $\sigma - W_2$ and $\|z\| < \varepsilon_0$. Choose an $\varepsilon_1 < \delta_1$ so that when $\|z\| < \varepsilon_1$, the set $\theta_z(\sigma)$ contains all of $\theta_0(\sigma - W_2)$. (This is possible; see 3.11 (b) of [6].) Then $\chi(x,z)$ is defined and belongs to σ when $\|z\| < \varepsilon_1$ and x is in $\sigma - W_2$.

Let $\gamma(x)$ be a C^∞ function on R^m such that $0 \le \gamma(x) \le 1$, $\gamma(x) = 1$ in a neighborhood of $\sigma - W_1$, and $\gamma(x) = 0$ in a neighborhood of $\overline{W}_2$ and outside σ.

Let $\beta(t)$ be a C^∞ real function such that $0 \le \beta(t) \le 1$, $\beta(t) = 1$ in a neighborhood of $t \le 0$ and $\beta(t) = 0$ in a neighborhood of $t \ge 1$.

Let $\alpha < \min(\varepsilon_0, \varepsilon_1)$; a further requirement on α is given later. We then may define

$$\chi_t(x,z) = x + t\beta(\|z\|/\alpha)\,\gamma(x)[\chi(x,z) - x],$$

for all (x,z) in $R^m \times R^p$, since the coefficient of $\chi(x,z) - x$ vanishes when χ is not defined. We define

$$f_t(x,z) = (\chi_t(x,z),z),$$

and check that if α is small, the map $f = f_1$ satisfies (a)–(c). To check this, we first compute

$$\frac{\partial \chi_t}{\partial x}(x,z) = I + t\beta\left(\frac{\|z\|}{\alpha}\right)\gamma(x)\left[\frac{\partial \chi}{\partial x}(x,z) - I\right] + t\beta\left(\frac{\|z\|}{\alpha}\right)\frac{\partial \gamma(x)}{\partial x}[\chi(x,z) - x],$$

where in the last term χ and x are written as row matrices. Both expressions in brackets are uniformly small when x is small, since the map $x \to \chi(x,z)$ is a good C^1 approximation to the identity when z is small. (θ_z is a good C^1 approximation to θ_0, so θ_z^{-1} is a good C^1 approximation to θ_0^{-1}, and $\chi(x,z) = \theta_z^{-1}\theta_0$ is a good C^1 approximation to the identity. See 3.7 of [6].) The other factors are bounded independent of α. Choose α small enough that $\partial\chi_t/\partial x$ is non-singular for all x and z.

For any z, the map $x \to \chi_t(x,z)$ carries $\bar{\sigma}$ into R^m; it equals the identity on Bd σ and is a local homeomorphism on σ. By 8.2 of [2], it is a homeomorphism of $\bar{\sigma}$ with itself. Hence f_t is a diffeomorphism of each closed cell of $\bar{\sigma} \times B^p$ with itself, for each t, so that (b) holds. Condition (a) is easy to check, but condition (c) requires some work.

Let c_δ be a transverse cell, and let x_0 be a point of $c_\delta \cap \sigma$. First, suppose x_0 is not in W_1. Then $\gamma(x) = 1$ in a neighborhood of x_0, so that $\chi_1(x, z) = \chi(x,z)$ for x in this neighborhood and z small. Hence in a neighborhood of $(x_0,0)$,

$$gf(x,z) = g(\chi(x,z),z) = (X(\chi,z),Z(\chi,z)) = (X(x,0),Z(\chi,z)).$$

If (x,z) lies in this neighborhood of $(x_0,0)$ and x lies in $c_\delta \cap \sigma$, then $gf(x,z)$ lies in $g_0(c_\delta \cap \sigma) \times R^p$, as desired.

Likewise, if x_0 is in $\overline{W}_2$ there is no problem, since $f(x,z)$ equals the identity in a neighborhood of $(x_0,0)$.

Finally, suppose x_0 is in $W_1 - \overline{W}_2$. We first show that there is a neighborhood Q of $(x_0,0)$ in $(c_\delta \cap \sigma) \times B^p$ which the map $(x,z) \to \chi(x,z)$ carries into $c_\delta \cap \sigma$. To verify this, note that the map $(x,z) \to (X(x,z), Z(x,z))$ carries some such neighborhood Q_1 into $g_0(c_\delta \cap \sigma) \times R^p$, by hypothesis. Then the map $(x,z) \to (X(x,z),z) = (\theta_z(x),z)$ also carries Q_1 into $g_0(c_\delta \cap \sigma) \times R^p$; if Q_1 is small it is a homeomorphism on Q_1. Since Q_1 and $g_0(c_\delta \cap \sigma) \times R^p$ are manifolds of the same dimension, the image of Q_1 under this map is necessarily open in $g_0(c_\delta \cap \sigma) \times R^p$. Hence the composite of $(x,z) \to (X(x,0),z)$ and the inverse of $(x,z) \to (X(x,z),z)$, that is, the map $(x,z) \to (\chi(x,z),z)$, carries some neighborhood Q of $(x_0,0)$ in $(c_\delta \cap \sigma) \times B^p$ into $(c_\delta \cap \sigma) \times B^p$, as desired.

To complete the proof, we note that *since $c_\delta \cap \sigma$ is convex*, the map $f(x,z) = (\chi_1(x,z),z)$ also carries Q into $(c_\delta \cap \sigma) \times B^p$. Now by hypothesis the map g carries a neighborhood V of $(x_0,0)$ in $(c_\delta \cap \sigma) \times B^p$ into $g_0(c_\delta \cap \sigma) \times R^p$. Since $f(x_0,0) = (x_0,0)$, the set $f^{-1}(Q \cap V)$ is a neighborhood of $(x_0,0)$ in $(c_\delta \cap \sigma) \times B^p$ which the composite gf carries into $g_0(c_\delta \cap \sigma) \times R^p$, as desired.

LEMMA 1.8. *Let $h : K \to M$ be a smooth triangulation; let $\delta > 0$. Let s be an n-simplex of K; let σ be a face of s of dimension m. Let u_σ and u_s be two coordinate systems about $h(\bar{\sigma})$ and $h(\bar{s})$ respectively, such that $\bar{\sigma}$ is rectilinear under $u_\sigma h$ and $\bar{s}$ is rectilinear under $u_s h$. Suppose that each transverse cell c_δ of K is orthogonal under $u_\sigma h$ and $u_s h$ to each proper face of σ, and that each c_δ is also orthogonal under $u_\sigma h$ to σ.*

Then given $\gamma < \delta$, there is a coordinate system u_s' about $h(\bar{s})$ such that $u_s'h(\sigma_0) = u_s h(\sigma_0)$ for each face σ_0 of s, and such that each c_γ is orthogonal to σ under $u_s'h$. The map u_s' equals u_s outside a given neighborhood of $h(\sigma)$, and also within some neighborhood of $h(Bd\ \sigma)$.

Proof. We consider R^m as the space consisting of those infinite sequences $(x^1,x^2, \ldots)$ for which $x^i = 0$ for $i > m$; thus R^m is contained naturally in R^n. We may assume that σ, $u_\sigma h(\sigma)$, and $u_s h(\sigma)$ all lie in R^m, for convenience.

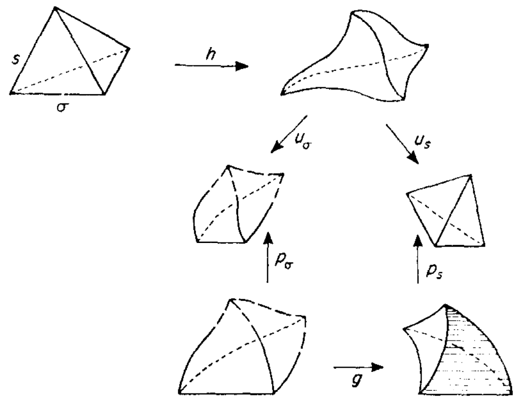

FIG. 1.

Let p_s be the trivial extension of $u_s h \mid \bar{\sigma}$ to an imbedding of $\bar{\sigma} \times R^p$ into $R^m \times R^p = R^n$, where $p = n - m$. Similarly, let p_σ be the trivial extension of $u_\sigma h \mid \bar{\sigma}$. Without change of notation, extend each to a neighborhood of $\bar{\sigma} \times R^p$. Then $p_\sigma^{-1} u_\sigma$ and $p_s^{-1} u_s$ are coordinate systems about $h(\bar{\sigma})$, and both $p_\sigma^{-1} u_\sigma h \mid \bar{\sigma}$ and $p_s^{-1} u_s h \mid \bar{\sigma}$ equal the identity. The fact that each c_δ is orthogonal to σ under $u_\sigma h$ implies it is also orthogonal to σ under $p_\sigma^{-1} u_\sigma h$, since p_σ preserves orthogonality of planes to R^m. From the hypothesis and 1.6, it follows that there is a neighborhood W_1 of *Bd* σ such that each cell c_γ is locally orthogonal to σ under $p_s^{-1} u_s h$ at points of $W_1 \cap c_\gamma \cap \sigma$.

Let g denote the map $(p_s^{-1} u_s)(p_\sigma^{-1} u_\sigma)^{-1}$. Then g is a diffeomorphism of a neighborhood of $\bar{\sigma}$ in R^n with another such neighborhood which 'equals the identity on $\bar{\sigma}$. The local orthogonality conditions imply that for each transverse cell c_γ and each y in $W_1 \cap c_\gamma \cap \sigma$, the following holds:

(*) *There is a neighborhood V of y in $(c_\gamma \cap \sigma) \times R^p$ such that $g(V)$ lies in $(c_\gamma \cap \sigma) \times R^p$.*

We will alter g, obtaining a new diffeomorphism g' whose domain and image are the same as those of g, such that

(a) *g' equals g outside a given neighborhood N of σ and in some neighborhood of Bd σ; it also equals g on σ.*

(b) *Condition* (*) *holds with g replaced by g', for each y in $c_\gamma \cap \sigma$.*

(c) *g' and g carry $p_\sigma^{-1} u_\sigma h(\sigma_0)$ onto the same set, for each face σ_0 of s.*

We will then define $u_s' = p_s g' p_\sigma^{-1} u_\sigma$ within the neighborhood $u_\sigma^{-1} p_\sigma(N)$ of $h(\sigma)$, and $u_s' = u_s$ otherwise; u_s' will clearly be a coordinate system about $h(\bar{s})$. It follows from (b) that each c_γ is orthogonal to σ under $u_s' h$. Condition (c) implies that $u_s' h$ carries each face σ_0 of s onto $u_s h(\sigma_0)$. The other conditions on u_s' are easy to check.

We assume N is small enough that it does not intersect $p_\sigma^{-1} u_\sigma h(\bar{s} - St\ \sigma)$.

Construction of g'. Let g be expressed in coordinates by $g(x,z) = (X(x,z),\ Z(x,z))$, where x and X are in R^m, and z and Z are in R^p. Let W_2, ε_0, $\beta(t)$, and $\gamma(x)$ be chosen as in the proof of the preceding lemma. Choose $\alpha < \varepsilon_0$; further conditions on α will be given later. Whenever (x,z) is in the domain of g, we define

$$X'(x,z) = X(x,z) + \beta(\|z\|/\alpha)\,\gamma(x)[x - X(x,z)],$$

and

$$g'(x,z) = (X'(x,z),\, Z(x,z)).$$

Clearly g' satisfies (a) above. To check (b), let x_0 be a point of $c_\gamma \cap \sigma$. If x_0 is not in W_1, then $\gamma(x) = 1$ in a neighborhood of x_0, so that $g'(x,z) = (x, Z(x,z))$ for x in this neighborhood and z small. Then (*) holds for x_0. If x_0 is in W_1, then by hypothesis there is a neighborhood V of x_0 in $(c_\gamma \cap \sigma) \times R^p$ such that $g(V)$ lies in $(c_\gamma \cap \sigma) \times R^p$. The map $(x,z) \to X(x,z)$ carries V into $c_\gamma \cap \sigma$; since $c_\gamma \cap \sigma$ is convex, so does the map $(x,z) \to X'(x,z)$. Hence g' carries V into $(c_\gamma \cap \sigma) \times R^p$.

Now we show that if α is small, g is a diffeomorphism. By direct computation, one shows that $\partial X'/\partial x(x,z)$ approximates $\partial X/\partial x(x,0)$ as closely as desired if α is small enough, and that $\partial X'/\partial z(x,z)$ is bounded independent of α. Since $\partial Z/\partial x(x,z) \to 0$ uniformly as $z \to 0$ for x in $\bar{\sigma}$, it follows that for any $a > 0$, there is an $\varepsilon_1(a) > 0$ such that det $(Dg'(x,z))$ approximates

$$\det(\partial X/\partial x(x,0))\,\det(\partial Z/\partial z(x,0)) = \det(Dg(x,0))$$

within a whenever $\alpha < \varepsilon_1$ and $\|z\| < \varepsilon_1$. We choose a less than the minimum of det $(Dg(x,0))$ on $\bar{\sigma}$, and we require that $\alpha < \varepsilon_1(a)$. Then g' is non-singular. Finally, choose ε_2 small enough that whenever x is in $\bar{\sigma}$ and $\|z\| \leq \varepsilon_2$, then (x,z) is in the domain of g; let C be the set of all such (x,z). Let $\alpha < \varepsilon_2$. Then g' maps the closed n-cell C into R^n; it agrees with the homeomorphism $g : C \to R^n$ on $Bd\, C$ and is a local homeomorphism at points of Int C. Hence g must be a homeomorphism of C onto $g(C)$, by 8.2 of [2].

Finally, we need to check (c). If σ_0 does not have σ as a face, this is trivial, since g' equals g on σ_0. Suppose σ_0 has σ as a face. Let Σ denote the rectilinear simplex $u_s h(\sigma)$; and Σ_0, the face $u_s h(\sigma_0)$. Note that $u_s h$ and $u_s' h$ are homeomorphisms of σ_0 into R^n which agree on $Bd\, \sigma_0$. If we can prove that $u_s' h$ carries σ_0 into the plane of Σ_0, it follows that $u_s' h(\sigma_0) = \Sigma_0$, since $u_s' h$ is a homeomorphism. Let (x,z) lie in $p_\sigma^{-1} u_{\bar{\sigma}} h(\sigma_0)$, and suppose that $g'(x,z) \neq g(x,z)$. Then by the definition of g', these two points have the same z-coordinate. As a result, the z-coordinates of $p_s g(x,z)$ and $p_s g'(x,z)$ are also the same. Since the former point lies in Σ_0, and since the plane of Σ_0 contains all of R^m, the latter point must lie in this plane as well.

§2. SMOOTHABILITY IMPLIES DEFORMABILITY

THEOREM 2.1. *Let $h' : K \to M$ and $k' : K \to N$ be smooth triangulations, let $\varepsilon > 0$, and let u_s and v_s be coordinate systems such that the transverse cells c_ε of K are orthogonal to the simplices of K under $u_s h'$ and $v_s k'$, as in 1.4. Let $f' = k'(h')^{-1}$. If f' may be smoothed to a diffeomorphism mod the subcomplex L of $h'(K)$, it may be smoothed to a diffeomorphism mod L which carries each dual cell $h'(\tau)$ onto $k'(\tau)$.*

Proof. What we prove is that the smoothing g of f' may so be chosen that g carries $h'(\bar{\tau}(\sigma))$ into $k'(c_\varepsilon(\sigma))$, for each dual cell $\tau(\sigma)$. It then follows by an induction argument that g carries $h'(\bar{\tau}(\sigma))$ onto $k'(\bar{\tau}(\sigma))$: Assuming it true for the faces of $\bar{\tau}$, g is a homeomorphism of the closed m-cell $h'(\bar{\tau})$ into the open m-cell $k'(c_\varepsilon)$, which carries $Bd\, h'(\bar{\tau})$ onto $Bd\, k'(\bar{\tau})$; one then applies 8.2 of [2].

Now any smoothing of f' may be obtained by a step-by-step process, redefining the map in small neighborhoods of the open $n-1$ simplices of $h'(K) - L$, then in small neighborhoods of the open $n-2$ simplices, and so on. (See Remark 3.2). It will suffice then to consider just one step of the process, in which we apply the construction given in §4 of [2] to a single simplex. Let $\phi : M \to N$ be a diffeomorphism mod J which equals f' on J; we assume as an induction hypothesis that for some choice $0 < \gamma < \delta < \varepsilon$, ϕ carries each transverse cell $h'(c_\gamma(\sigma_0))$ into $k'(c_\delta(\sigma_0))$. Let σ be an m-simplex of $h^{-1}(J)$ which is the face of no other simplex of $h^{-1}(J)$; let the obstruction coefficient of ϕ vanish on σ so that the smoothing process given in Case I of 4.1 of [2] may be carried out. We show that the smoothed map satisfies this induction hypothesis as well.

Choose $\gamma_0 < \gamma_1 < \gamma$ and $\delta < \delta_0 < \delta_1 < \varepsilon$; choose an n-simplex s of K having σ as a face. We assume σ, $u_s h'(\sigma)$, and $v_s k'(\sigma)$ lie in the plane R^m of the first m coordinates. Let p_s be obtained by extending $u_s h' \mid \bar{\sigma} : \bar{\sigma} \to R^m$ to a neighborhood of $\bar{\sigma}$ in R^m, and then extending trivially to a diffeomorphism of a neighborhood of $\bar{\sigma}$ in $R^m \times R^{n-m} = R^n$ onto a neighborhood in R^n. Similarly, let q_s be an extension of $v_s k' \mid \bar{\sigma}$. We use $p_s^{-1} u_s$ and $q_s^{-1} v_s$ as coordinate neighborhoods of $h'(\bar{\sigma})$ and $k'(\bar{\sigma})$, respectively. Referring ϕ to these coordinate systems, we have the map

$$\psi = (q_s^{-1} v_s)\phi(p_s^{-1} u_s)^{-1},$$

which carries a neighborhood of $\bar{\sigma}$ in R^n into R^n, and equals the identity on $\bar{\sigma}$. We apply 4.1 of [2] to the map ψ.

For the moment, consider a single family $c_\alpha = c_\alpha(\sigma_0)$, where σ_0 is fixed and $\alpha < \varepsilon$. Since p_s preserves orthogonality of planes to R^m, each c_α is orthogonal to σ under the map $p_s^{-1} u_s h'$. In particular, this map carries a neighborhood of $c_\gamma \cap \sigma$ in c_γ onto an open set W in $(c_\gamma \cap \sigma) \times R^p$. Because $\bar{c}_{\gamma_1} \subset c_\gamma$, there is a neighborhood U_1 of σ in R^n whose intersection with $(c_{\gamma_1} \cap \sigma) \times R^p$ lies in this open set W, so that

(a) $U_1 \cap ((c_{\gamma_1} \cap \sigma) \times R^p)$ lies in $p_s^{-1} u_s h'(c_\gamma)$.

Similarly, if U_1 is chosen small enough, then

(b) $U_1 \cap (p_s^{-1} u_s h'(c_{\gamma_0}))$ lies in $(c_{\gamma_1} \cap \sigma) \times R^p$.

Likewise, there is a neighborhood U_2 of σ in R^n such that

(c) $U_2 \cap ((c_{\delta_0} \cap \sigma) \times R^p)$ lies in $q_s^{-1} v_s k'(c_{\delta_1})$, and

(d) $U_2 \cap (q_s^{-1} v_s k'(c_\delta))$ lies in $(c_{\delta_0} \cap \sigma) \times R^p$.

Now conditions (a)–(d) are stated for a single family of transverse cells, those transverse to σ_0. We require U_1 and U_2 to be small enough that (a)–(d) hold for each such family which intersects σ.

Let $\psi(x,z) = (X(x,z), Z(x,z))$, where x and X are in R^m and z and Z are in R^p. We also choose U_1 small enough that any map f_a, where $f_a(x,z) = (ax + (1 \quad a)X(x,z), Z(x,z))$ and $0 \leq a \leq 1$, carries U_1 into U_2.

The construction in 4.1 of [2] is carried out by altering ψ within U_1. We use the following fact, which is a consequence of (a) and (d):

(*) ψ carries $U_1 \cap ((c_{\gamma_1} \cap \sigma) \times R^p)$ into $U_2 \cap ((c_{\delta_0} \cap \sigma) \times R^r)$.

The *first modification* of ψ(4.2 of [2]) replaces ψ by a map of the form $\psi_1(x,z) = (X_1(x,z), Z(x,z))$, where X_1 is on the line segment between x and $X(x,z)$. (The x, z-notation is reversed from that given in 4.2 of [2], but this should cause no difficulty.) Because $(c_{\delta_0} \cap \sigma)$ is convex and f_a carries U_1 into U_2, (*) will still be satisfied if we replace ψ by ψ_1. All further modifications leave the X coordinate strictly alone, altering only the Z coordinate, and carrying U_1 onto $\psi_1(U_1)$; thus (*) is also satisfied by the map ψ' finally obtained.

Conditions (b) and (c) above combine with (*) to show that the resulting map $\phi' = (q_s^{-1}v_s)^{-1}\,\psi'(p_s^{-1}u_s)$ will carry $h'(c_{\gamma_0})$ into $k'(c_{\delta_1})$, as desired.

THEOREM 2.2. *Let $f: M \to N$ be a combinatorial equivalence. If f may be smoothed to a diffeomorphism mod L, then given any neighborhood U of L, f may be combinatorially deformed to a combinatorial equivalence $g: M \to N$ which is a diffeomorphism outside U. The converse holds as well; the theorem also holds if the combinatorial deformation is required to be strong* (*i.e. level-preserving*).

Proof. We first prove the theorem; the converse follows directly from [5] and is saved for later. Remark 3.3 shows that subdividing K does not affect the smoothability of f; nor does it affect the conclusion of the theorem. Hence we may assume, without change of notation, that $St\, J$ lies in $h^{-1}(U)$, where $J = h^{-1}(L)$. By 1.4, we alter h and k by smooth isotopies to h' and k', so that each transverse cell c_ε of K is orthogonal to each simplex σ of K under $u_s h'$ and $v_s k'$, where u_s and v_s are suitably chosen coordinate systems about $h'(\bar{s})$ and $k'(\bar{s})$, respectively. This alters f by a combinatorial deformation to f'; by 3.1, f' is smoothable to a diffeomorphism mod $L = h'(J)$.

For each dual m-cell τ of K, let $N_\tau = h'(c_\varepsilon)$ be the smooth submanifold of M containing it, as in 1.5. Let K_0 be that subcomplex of the first barycentric subdivision K' of K whose polytope is the union of those closed dual cells $\bar{\tau}$ of K which do not intersect J. Let $M_0 = h'(K_0)$.

By 2.1, there exists a homeomorphism $g': M \to N$ which carries $h'(\tau)$ onto $k'(\tau)$ for each τ, such that $g' \mid (M - L)$ is a diffeomorphism. Let $h_0' = h' \mid K_0$ and let $\ell_0 = (g')^{-1}k' \mid K_0$; each is a smooth imbedding of K_0 in M. Both h_0' and ℓ_0' carry τ into N_τ, for each τ. By 5.3 of [5] or 10.11 of [6], given any $\delta > 0$, there are subdivisions K_0' and K_0'' of K_0, and δ-approximations $h_0'': K_0' \to M$ and $\ell_0'': K_0' \to M$ to h_0' and ℓ_0', respectively, which are imbeddings, which intersect in a full subcomplex, and whose union is an imbedding; they may be chosen so that they carry each dual cell $\tau \subset K_0$ into the submanifold N_τ. (The lemma quoted is stated only for a finite number of transversally intersecting non-bounded manifolds, but it is proved just as easily for a collection which is locally-finite.) The usual induction argument shows that they must carry each τ onto $h'(\tau)$, so that in particular they carry $|K_0|$ onto M_0. The composite $(\ell_0'')^{-1}h_0''$ is thus a piecewise-linear homeomorphism ϕ_0 of K_0 with itself preserving dual cells; let h_0''' be the piecewise-smooth triangulation $\ell_0'(\ell_0'')^{-1}h_0'': K_0 \to M_0$.

Given $\delta_1 > 0$, the above δ may be chosen small enough that h_0''' is a δ_1-approximation to h_0'. Applying Remark 3.4, one finds that given $\delta_2 > 0$, δ_1 in turn may be chosen small enough that any δ_1-approximation $h_0''': K_0''' \to M$ to h_0' may be extended to a δ_2-approximation $h''': K''' \to M$ to h' which equals h' on $|J|$ and carries each set τ into N_τ. δ_2 may in turn be chosen small enough that h''' is a triangulation; by the usual argument, $h'''(\tau) = h'(\tau)$ for each τ. Finally, δ_2 may also be chosen small enough that h''' is smoothly isotopic to $h': K''' \to M$; the isotopy will equal the trivial extension of h' on J (10.9 of [6]).

Now ϕ_0 is a piecewise-linear homeomorphism of K_0 with itself which carries each cell τ onto intself. A simple induction argument shows that ϕ_0 may be extended to a piecewise-linear homeomorphism ϕ of K with itself which preserves dual cells; Suppose ϕ_0 is extended to all the dual $m - 1$ cells of K. We note that each dual m-cell $\bar{\tau}(\sigma)$ is the join of a combinatorial sphere with the point $\hat{\sigma}$, and we extend the given piecewise-linear homeomorphism of $Bd\,\tau$ with itself radially to a piecewise-linear homeomorphism of $\bar{\tau}$ with itself. (One may check that ϕ automatically equals the identity on J.) The same argument is used to show that ϕ is piecewise-linearly isotopic to the identity; the isotopy is equal to the identity on $J \times I$.

Let us consider the piecewise smooth triangulations $h'''\phi^{-1} : K \to M$ and $k' : K \to N$. Let g be the composite $k'\phi(h''')^{-1} : M \to N$. Then g is a combinatorial equivalence of M with N. It is a diffeomorphism outside U, for when restricted to M_0 (which contains $M - U$), g equals

$$k'[(\ell_0'')^{-1}h_0''](h_0''')^{-1} = k'[(\ell_0')^{-1}h_0'''](h_0''')^{-1} = g'.$$

Since ϕ is piecewise-linearly isotopic to the identity, and h''' is smoothly isotopic to h', $f' = k'(h')^{-1}$ is strongly combinatorially deformable to g. Since f is strongly combinatorially deformable to f', the theorem is proved.

Proof of the converse. We prove a slighly strengthened converse. Instead of requiring the deformation to exist for arbitrarily small neighborhoods U of L, we shall only assume that it exists for some U such that $h^{-1}(\bar{U})$ lies in a polyhedron P in K for which the inclusion $J \to P$ is a homotopy equivalence, where $J = h^{-1}(L)$. Since subdividing affects neither hypothesis nor conclusion, we may assume without change of notation that P is a subcomplex of K. Let $(\tilde{h}, \tilde{k}, K \times I)$ be a combinatorial deformation between f and g, where g is a diffeomorphism outside U. Then $\tilde{h}\ (P \times 0)$ contains $U \times 0$.

We construct a combinatorial deformation $\tilde{h}'$, $\tilde{k}'$ between f and g such that $\tilde{h}'(P \times 1)$ contains $U \times 1$: Let $\tilde{h}' : (K \times I)' \to M \times I$ be a δ-approximation to $(h \times i) : (K \times I) \to M \times I$ which equals h on $K \times 0$, such that $\tilde{h}^{-1}\tilde{h}'$ is piecewise-linear. We may assume that $(K \times 0)'$ equals $K \times 0$; choose δ small enough that $\tilde{h}'(P \times I)$ contains $U \times I$. Let $\tilde{k}' = \tilde{k}(\tilde{h})^{-1}\tilde{h}'$; then $\tilde{h}'$, $\tilde{k}'$ is the desired combinatorial deformation between f and g.

Extend $\tilde{h}'$ and $\tilde{k}'$ trivially to triangulations $(K \times R)' \to M \times R$ and $(K \times R)' \to N \times R$ respectively. They define a combinatorial equivalence $F : M \times R \to N \times R$; F is also a deformation mod

$$\tilde{h}'((K \times (-\infty,0]) \cup (K \times I)' \cup (P \times [1,\infty))')$$

between f and g, since g is a diffeomorphism on $M - P$. We apply the argument given in 3.1 to conclude that f may be smoothed to a diffeomorphism f' mod $\tilde{h}'(P \times 0)) = h(P)$. Then since the inclusion $J \to P$ is a homotopy equivalence, we apply 5.1 of [5] to conclude that f' may in turn be smoothed to a diffeomorphism mod $h(J) = L$.

REMARK 2.3. *Theorem 2.2 also holds if the conclusion is strengthened to require that the combinatorial deformation equal the trivial extension of h, k on the subcomplex $J = h^{-1}(L)$.* All we know about the deformation $\tilde{h}$, $\tilde{k}$ constructed in the proof is that it is strong and that it carries each set $\sigma \times I$ into $h(\sigma) \times I$ and $k(\sigma) \times I$, respectively, for each σ in J. (For during the deformation of f to f', we altered h and k on J.) But it is just an exercise in using the techniques of J. H. C. Whitehead to modify $\tilde{h}$ and $\tilde{k}$ so as to satisfy this additional condition.

COROLLARY 2.4. *Let $h : K \to M$ and $k : K \to N$ be smooth triangulations; let $f = kh^{-1}$. Let h' and k' be triangulations smoothly isotopic to h and k, respectively, satisfying the hypotheses of* 2.1. *Let α and β be the differentiable structures on K induced by h and k, respectively. Then the following statements are equivalent:*

(0) *There exists a diffeomorphism $g : M \to N$ carrying each dual cell $h'(\tau)$ onto $k'(\tau)$.*

(1) *f is smoothable to a diffeomorphism.*

(1a) *f is smoothable to a diffeomorphism g satisfying* (0).

(2) *f is combinatorially deformable to a diffeomorphism.*

(2a) *f is strongly combinatorially deformable to a diffeomorphism g satisfying* (0); *the deformation $(\tilde{h}, \tilde{k})$ may be chosen so that $\tilde{h}$ is the trivial extension $h \times i$ of h.*

(3) *α is concordant to β.*

(3a) *There is a concordance γ between α and β such that each slice $K \times t$ is a differentiable submanifold of $(K \times I)_\gamma$.*

Proof. Part of the proof was given in the introduction. Only the implications (1) ⇒ (0) and (0) ⇒ (2a) remain to be proved. The fact that (1) ⇒ (0) is the substance of Theorem 2.1. f is combinatorially deformable to $f' = k'(h')^{-1}$, so that f' may be smoothed to a diffeomorphism; then 2.1 implies that f' may be smoothed to a diffeomorphism g satisfying (0).

The implication (0) ⇒ (2a) follows from the proof of Theorem 2.2 : We take U and L to be empty, so that no subdivision of K is required. The proof then proceeds unchanged; it constructs a strong combinatorial deformation $(\tilde{h}, \tilde{k})$ between f and the specific diffeomorphism g hypothesized in (0). Some additional work is required to obtain the extra condition on the combinatorial deformation. We show that it may be chosen so that $\tilde{k} = k \times i$; the other alternative follows by symmetry.

Let $\tilde{h}(x,t) = (h_t(x),t)$ and $\tilde{k}(x,t) = (k_t(x),t)$. Then $h_0 = h$ and $h_1 = h'''\phi^{-1}$, while $k_0 = k$ and $k_1 = k'$ (assuming the notation used in 2.2). The map $\tilde{h} : K \times I \to M \times I$ is only piecewise smooth; but $\tilde{k} : K \times I \to N \times I$ is the smooth isotopy given by 1.4, so that $\tilde{k}$ is smooth on each set $\bar{\sigma} \times I$ and carries it onto $k(\bar{\sigma}) \times I$. Define $\tilde{H} : K \times [0,2] \to M \times [0,2]$ by the equations

$$\begin{aligned} \tilde{H}(x,t) &= \tilde{h}(x,t) = (h_t(x),t) && \text{for} \quad 0 \le t \le 1 \\ &= (h_1(k_{t-1})^{-1}k(x),t) && \text{for} \quad 1 \le t \le 2. \end{aligned}$$

We need only to show that $\tilde{H}$ is piecewise smooth; the theorem will follow, since $(\tilde{H},k \times i)$ will then be a strong combinatorial deformation between f and g. $\tilde{H}$ is clearly piecewise smooth on $K \times [0,1]$. Furthermore, h_1 is smooth on each simplex of K, since k_1 is and $h_1 = g^{-1}k_1$ where g is a diffeomorphism. Then $h_1(k_{t-1})^{-1}k$ is smooth on each set $\bar{\sigma} \times [1,2]$, so that $\tilde{H}$ is also smooth on this set. Hence $\tilde{H}$ is piecewise smooth on $K \times [0,2]$.

COROLLARY 2.5. *Let $h : K \to M$ and $k : K \to N$ be smooth triangulations; let $f = kh^{-1}$. Let α and β be the differentiable structures on K induced by h and k, respectively. Then the following are equivalent:*

(1) *f may be smoothed to a diffeomorphism mod the m-skeleton $h(K^m)$ of M.*

(2) *The differentiable structures α and β on $K \times 0$ and $K \times 1$, respectively, may be extended to a differentiable structure γ on a neighborhood in $K \times I$ of $(K \times 0) \cup (K^{p-1} \times I) \cup (K \times 1)$, where $p = n - m$.*

Proof. Let K' be the first barycentric subdivision of K. We need to distinguish here between the combinatorial equivalences (K,h,k) and (K',h,k); let us denote them by f and f', respectively. Let L be the image under h of the m-skeleton of the dual cell subdivision of K; L is a subcomplex of $h(K')$.

We prove first that (1) is equivalent to the statement:

(1a) *f' may be smoothed to a diffeomorphism mod L.*

By 3.2 of [5], the obstructions to smoothing f vanish in dimensions greater than m if and only if the obstructions to smoothing f' vanish in these dimensions. Hence (1) implies that f' may be smoothed to a diffeomorphism f'_m mod $h((K')^m)$. Now any m-chain of $h(K')$ is homologous to a chain carried by L; hence we may choose f'_m so that the obstruction chain $\lambda_m f'_m$ is carried by L (using 5.3 of [2]). Now $\mathscr{H}_j((K')^m, L) = 0$ for $j < m$ (with any coefficients) and $\lambda_m f'_m$ is zero on $K' - L$; it follows from the proof of 5.1 of [5] that f'_m may be smoothed to a diffeomorphism mod L (see the induction hypothesis of that proof). Conversely, if (1a) holds, then all the obstructions to smoothing f' (and hence f as well) vanish in dimensions greater than m (see 2.5 of [5]).

(1a) $\Rightarrow$ (2). Choose a neighborhood U of L such that U does not intersect $h(K^{p-1})$. By 2.2, there is a combinatorial deformation $F : M \times I \to N \times I$ between f and a combinatorial equivalence g such that g is a diffeomorphism outside U; we may assume F equals the trivial extension of g near $M \times 1$ and of f near $M \times 0$. The differentiable structure γ' on $K \times I$ induced by $F^{-1}(k \times i)$ equals α near $K \times 0$, and it equals β on the complement in $K \times 1$ of $h^{-1}(\bar{U})$. In particular, it equals β in a neighborhood of $K^{p-1} \times 1$. We define γ equal to γ' in a neighborhood of $(K \times 0) \cup (K^{p-1} \times I)$ and equal to β in a neighborhood of $K \times 1$.

(2) $\Rightarrow$ (1a). Conversely, suppose γ exists. Let γ' be the restriction of γ to a neighborhood of $(K \times 0) \cup (K^{p-1} \times I)$. A relative version of Hirsch's basic theorem (3.1 of [1]) implies that γ' may be extended to a differentiable structure δ on $K \times I$. The proof will appear in a forthcoming paper by Hirsch and B. Mazur.

As in the proof in Corollary 2.4 that (3) $\Rightarrow$ (2), there exists a diffeomorphism $G : (K \times I)_\delta \to K_\alpha \times I$ which equals the identity near $(K \times 0)_\delta$. Then $(h \times i)G : K' \times I \to M \times I$ and $(k \times i) : K' \times I \to N \times I$ constitute a combinatorial deformation between f' and a combinatorial equivalence g which is a diffeomorphism in a neighborhood V of $h(K^{p-1})$. There is a polyhedron P in K containing $M - V$ such that the inclusion $h^{-1}(L) \to P$ is a homotopy equivalance. Our result now follows from the converse of 2.2. (Compare the hypothesis stated at the beginning o. .he proof of this converse.)

§3. APPENDIX

The following remarks follow from the techniques of [5] and [6], and are collected here for convenience.

REMARK 3.1. Consider the definition of *smoothing* which was given in 1.4 of [5]. *Let $f : M \to N$ be a diffeomorphism mod L; let $f_1 : M \to N$ be a diffeomorphism mod a subdivision L_1' of a subcomplex L_1 of L; suppose there exists a deformation $F : M \times R \to N \times R$ mod some subdivision $\mathscr{L}'$ of $\mathscr{L} = (L \times R^-) \cup (L_1 \times R^+)$ between f and f_1.* The existence of F does not suffice for f_1 to be called a smoothing of f, for our definition requires that $L_1' = L_1$, that $\mathscr{L}' = \mathscr{L}$, and that F equal the trivial extension of f on all of $L_1 \times R$. However, *the existence of F does suffice to prove that there is a smoothing g of f which is a diffeomorphism mod L_1.*

The proof is a simple adaptation of the proof of 1.6 of [5]. Choose κ so that F is trivial below the level $-\kappa + 1$ (i.e., on $M \times (-\infty, \ -\kappa + 1])$. Now $\mathscr{J} = (L \times (-\infty, \ -\kappa]) \cup (L_1 \times [-\kappa, \ \infty))$ is a deformation retract of $\mathscr{L}$, so that 5.1 of [5] applies and F may be smoothed to a diffeomorphism G mod $\mathscr{J}'$. (Here $\mathscr{J}'$ is the subdivision of $\mathscr{J}$ induced by $\mathscr{L}'$. It equals $\mathscr{J}$ below the level $-\kappa + 1$.) Now G equals F on $\mathscr{J}$ itself, so that in particular G is the trivial extension of f on $L_1 \times (-\infty, \ -\kappa + 1]$. Furthermore, if one analyzes the construction of G, one finds it may be chosen so that it is level-preserving in a neighborhood of the slice $M \times (-\kappa + \frac{1}{2})$; this was noted (and was crucial) in the proof of 1.6 of [5]. The map g we want is the restriction of G to the slice $M \times (-\kappa + \frac{1}{2})$; to show it is a smoothing of f, we merely chop G off above the level $(-\kappa + \frac{1}{2})$ and stretch $[-\kappa, \ -\kappa + \frac{1}{2})$ out to fill $[-\kappa, \ \infty)$, just as in 1.6 of [5]. The new map H we obtain will be a deformation mod $(L \times (-\infty, -\kappa]) \cup (L_1 \times [-\kappa, \ \infty))$ between f and g which equals the trivial extension of f on $L_1 \times R$. Shifting H upwards by κ gives us our desired result.

REMARK 3.2. The following is a slight strengthening of Theorem 1.8 of [5] : *Let $f : M \to N$ be a diffeomorphism mod L, where L has dimension m. Suppose f may be smoothed to a diffeomorphism f' mod L_1, a subcomplex of L. Then there is a sequence of maps f_m, $f_{m-1}, \ldots, f_0, f_{-1}$ where $f_m = f$ and $f_{-1} = f'$, such that (for each j) f_j is a diffeomorphism mod*

$L^j \cup L_1$, and f_{j-1} is a smoothing of f_j which equals f_j outside an arbitrarily small neighborhood of the open j-simplices of $L - L_1$.

To prove this, one first generalizes Theorem 1.6 of [5] to prove that if $\lambda_m f$ is carried by L_1, then there exists a smoothing of f which is a diffeomorphism mod $L^{m-1} \cup L_1$. The proof given in [5] generalizes immediately; one merely replaces L^{m-1} throughout by $L^{m-1} \cup L_1$. One then generalizes Lemma 1.7 of [5], replacing L^{m-1} by $L^{m-1} \cup L_1$ and concluding that $d_m H$, which is defined on m-cells of $L - L_1$, agrees with c_m there.

After these preliminaries, the proof of 1.8 given in [5] goes through without difficulty. L^{m-1} is replaced by $L^{m-1} \cup L_1$ throughout, and $d_m F$ and $d_m H$ are defined only on $L - L_1$.

REMARK 3.3. *Let $f: M \to N$ be the combinatorial equivalence (K,h,k). Let K' be a subdivision of K, and let f' denote the combinatorial equivalence (K',h,k). Then f may be smoothed to a diffeomorphism mod L_1 if and only if f' may be smoothed to a diffeomorphism mod L_1'.* (Here L_1 is the image under h of a subcomplex of K, and L_1' is the subdivision induced by K'.)

Proof. Clearly there is a combinatorial deformation between f and f'. If f' may be smoothed to g, then there is a composite deformation connecting f to g which satisfies the hypotheses of 3.1, with $L = h(K)$. The converse is easier: if F is the hypothesized deformation mod $(L \times R^-) \cup (L_1 \times R^+)$ between f and g, it is also the desired deformation mod $(L' \times R^-) \cup (L_1' \times R^+)$.

REMARK 3.4. The following is a strengthening of 9.8 of [6]: *Let $f: K \to M$ be a smooth map; M non-bounded. Let K_1 be a subcomplex of K. Given $\varepsilon(x) > 0$, there is a $\delta(x) > 0$ such that for any δ-approximation g_1 to $f | K_1$, there is an ε-approximation $g : K''' \to M$ to f such that g equals g_1 on $|K_1|$ and equals f outside $St(K_1,K)$; the subdivision K''' of K may be chosen to equal K outside $St\,(K_1,K)$.*

Let $\{N_i\}$ be a locally-finite collection of non-bounded transversally intersecting smooth submanifolds of M; let $\{J_i\}$ be a collection of subcomplexes of K such that, for each i, $f(J_i)$ is contained in N_i and is closed in M. If $g_1(J_i \cap K_1) \subset N_i$ for each i, then we may choose g so that $g(J_i) \subset N_i$ for all i.

Proof. Each point of $f(J_i)$ has a coordinate neighborhood $h : U \to R^n$ such that for each j, $h(U \cap N_j)$ is empty or equals $h(U) \cap P_j$, where P_j is a plane in R^n whose dimension is that of N_j. Cover each set $f(J_i)$ by such coordinate neighborhoods; cover the complement of the closed set $\bigcup_i f(J_i)$ by coordinate neighborhoods; pass, by shrinking, to a locally-finite covering of M. Subdivide K fine enough to K' that for each simplex σ of K', $f(\bar{\sigma})$ lies in one of these coordinate neighborhoods; let ε be small enough that $g(\bar{\sigma})$ necessarily lies in this coordinate neighborhood as well. Let K' be fine enough that $St^2(|K_1|,K')$ lies in $St(|K_1|,K)$, also.

Now apply 9.8 and 9.8(c) of [6] to each simplex of K' lying in $St(|K_1|,K') - |K_1|$ in turn, working up in dimension, and obtain an extension of g_1. At the end of the process, one has a smooth map $g : K'' \to M$, where K'' is a subdivision of K'. Let us choose finally a subdivision K''' of K which equals K'' on the closure of $St(K_1,K')$ and equals K outside $St^2(K_1,K')$. Since g equals f outside $St(K_1,K')$, $g : K''' \to M$ is a smooth map, as desired.

REFERENCES

1. M. Hirsch: Obstruction theories for smoothing manifolds and maps, *Bull. Am. Math. Soc.* **69** (1963), 352–356.
2. J. R. Munkres: Obstructions to the smoothing of piecewise-differentiable homeomorphisms, *Ann. Math.* **72** (1960), 521–554.
3. J. R. Munkres: Obstructions to extending diffeomorphisms, *Proc. Am. Math. Soc.* **15** (1964), 297–299.
4. J. R. Munkres: Obstructions to imposing differentiable structures, *Illinois J. of Math.* **8** (1964), 361–376.
5. J. R. Munkres: Higher obstructions to smoothing, *Topology* **4** (1965), 27–45.
6. J. R. Munkres: Elementary differential topology, (revised edition), *Ann. Math. Study* No. 54, Princeton Univ. Press, 1966.

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